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Integral Representations for Jacobi Polynomials and Some Applications*

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To Professor Gábor Szegő on his 75th birthday

An integral for $[P_n^{(\alpha+\mu, \beta)}(x)]/[P_n^{(\alpha+\mu, \beta)}(1)]$ in terms of $[P_n^{(\alpha, \beta)}(y)]/[P_n^{(\alpha, \beta)}(1)]$ with a positive kernel is obtained. For $\beta = \pm \frac{1}{2}$ this integral is equivalent to an important integral of Feldheim and Vilenkin connecting ultraspherical polynomials. As an application we show that

$$\frac{P_n^{(\alpha, \alpha)}(x)}{P_n^{(\alpha, \alpha)}(1)} = \int_{-1}^1 \frac{P_n^{(\beta, \beta)}(y)}{P_n^{(\beta, \beta)}(1)} d\mu(y)$$

where $\alpha > \beta \geq -\frac{1}{2}$, $-1 \leq x \leq 1$, and $d\mu(y)$ is a positive measure which depends on x but not n . For $\beta = -\frac{1}{2}$ this is a result of Seidel and Szasz. Similar results are obtained for Jacobi polynomials and the positivity of certain sums of ultraspherical and Jacobi polynomials is obtained.

1. INTRODUCTION

In studying special functions one of the first questions that is asked is whether you can obtain an explicit formula for the function in question, either as a series or an integral of some more elementary functions. For instance, the Legendre polynomial $P_n(x)$ can be written as

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \Gamma(n-k+\frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(k+1) \Gamma(n-2k+1)} (2x)^{n-2k}, \quad [30, (4.7.31)], \quad (1.1)$$

which is as explicit an expression as one can hope for and the functions x^k that are used in the expansion are the most elementary of all functions. Unfortun-

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nately (1.1) is not a useful representation for many problems. For instance, $P_n(x)$ satisfies

$$|P_n(x)| \leq 1, \quad -1 \leq x \leq 1 \quad (1.2)$$

and (1.2) is far from obvious from the representation (1.1).

There is another series expansion for $P_n(x)$ from which (1.2) follows immediately.

$$P_n(\cos \theta) = \sum_{k=0}^n \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{n-k} \cos(n-2k)\theta \quad (1.3)$$

where

$$(\alpha)_n = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}. \quad (1.4)$$

$P_n(x)$ is normalized by $P_n(1) = 1$ and $\left(\frac{1}{2}\right)_k > 0$ so

$$|P_n(\cos \theta)| \leq \sum_{k=0}^n \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{n-k} = P_n(1) = 1.$$

This expansion is a very useful one and it shows why it is useful to have expansions with positive coefficients; or more generally, in integral expansions it is useful to have a positive kernel. Often we are even willing to pay the price of expanding in terms of a more complicated set of functions, if at the same time we have a positive kernel.

In this paper we will be concerned with Jacobi polynomials and our expansions will be series and integral expansions of Jacobi polynomials in terms of other Jacobi polynomials. The common feature of all of the expansions will be the positive kernel that occurs.

$P_n^{(\alpha, \beta)}(x)$, the Jacobi polynomial of degree n , order (α, β) , is defined by

$$P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} [(1-x)^{n+\alpha} (1+x)^{n+\beta}], \quad \alpha, \beta > -1. \quad (1.5)$$

As special cases of $P_n^{(\alpha, \beta)}(x)$ we have the following.

$$\frac{P_n^{(-\frac{1}{2}, -\frac{1}{2})}(\cos \theta)}{P_n^{(-\frac{1}{2}, -\frac{1}{2})}(1)} = \cos n\theta \quad (1.6)$$

$$\frac{P_n^{(\frac{1}{2}, \frac{1}{2})}(\cos \theta)}{P_n^{(\frac{1}{2}, \frac{1}{2})}(1)} = \frac{\sin(n+1)\theta}{(n+1)\sin \theta} \quad (1.7)$$

$$\frac{P_n^{(\frac{1}{2}, -\frac{1}{2})}(\cos \theta)}{P_n^{(\frac{1}{2}, -\frac{1}{2})}(1)} = \frac{\sin(n+\frac{1}{2})\theta}{(2n+1)\sin(\theta/2)} \quad (1.8)$$

$$\frac{P_n^{(-\frac{1}{2}, \frac{1}{2})}(\cos \theta)}{P_n^{(-\frac{1}{2}, \frac{1}{2})}(1)} = \frac{\cos(n+\frac{1}{2})\theta}{\cos(\theta/2)}. \quad (1.9)$$

The value of $P_n^{(\alpha, \beta)}(1)$ has commonly been standardized to

$$P_n^{(\alpha, \beta)}(1) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1) \Gamma(\alpha + 1)} = \frac{(\alpha + 1)_n}{n!}. \quad (1.10)$$

All of these formulas, as well as any other results that we give without a specific reference, are in Chapter 4 of Szegő's book "Orthogonal Polynomials" [30].

One other very interesting and useful special case is the Legendre case, $\alpha = \beta = 0$. From (1.6) we see that (1.3) is an expansion of the form

$$P_n^{(\gamma, \delta)}(x) = \sum_{k=0}^n \alpha_k P_k^{(\alpha, \beta)}(x) \quad (1.11)$$

where $\gamma = \delta = 0$, $\alpha = \beta = -\frac{1}{2}$. When $\alpha = \beta = -\frac{1}{2}$ and $\gamma = \delta > -\frac{1}{2}$ (1.11) is well known and is given in most of the books that give formulas for special functions. In this case α_k is the product of gamma functions and is positive. There are other special cases of (1.11) known with α_k given by gamma functions [1], and in the general case α_k is a ${}_3F_2$, [16], [2]. We will give a summary of these results later, as we have some new applications of them.

In connection with integral representations we first mention a trivial integral which has a generalization to Jacobi polynomials which is of no use to us.

$$\frac{\sin n\theta}{n} = \int_0^\theta \cos n\varphi \, d\varphi. \quad (1.12)$$

If we use (1.6) and (1.7) we see that (1.12) is

$$\frac{P_{n-1}^{(\frac{1}{2}, \frac{1}{2})}(\cos \varphi)}{P_{n-1}^{(\frac{1}{2}, \frac{1}{2})}(1)} \sin \varphi = \int_0^\varphi \frac{P_n^{(-\frac{1}{2}, -\frac{1}{2})}(\cos \theta)}{P_n^{(-\frac{1}{2}, -\frac{1}{2})}(1)} d\theta \quad (1.13)$$

or equivalently

$$\frac{P_{n-1}^{(\frac{1}{2}, \frac{1}{2})}(x)}{P_{n-1}^{(\frac{1}{2}, \frac{1}{2})}(1)} (1 - x^2)^{\frac{1}{2}} = \int_x^1 \frac{P_n^{(-\frac{1}{2}, -\frac{1}{2})}(y)}{P_n^{(-\frac{1}{2}, -\frac{1}{2})}(1)} (1 - y^2)^{-\frac{1}{2}} dy. \quad (1.14)$$

(1.14) is the special case $\alpha = \beta = -\frac{1}{2}$, $m = 1$ of

$$\begin{aligned} & (1 - x)^\alpha (1 + x)^\beta P_n^{(\alpha, \beta)}(x) \\ &= \frac{(-1)^m (n - m)!}{2^m n!} \left(\frac{d}{dx} \right)^m \{ (1 - x)^{m+\alpha} (1 + x)^{m+\beta} P_{n-m}^{(\alpha+m, \beta+m)}(x) \}, \end{aligned} \quad (1.14')$$

$n \geq m$. This is (4.10.1) in [30]. This is useful, and in particular it contains Rodrigues' formula. But the type of applications which can be made of (1.14') are different than the applications we will consider. Also it doesn't have a fractional integral type generalization, because the degree of $P_n^{(\alpha, \beta)}(x)$ changes under the operation. There are other formulas for Jacobi polynomials which have P_n on both sides and these have fractional integral generalizations. Consider, for example, the functions which occur in (1.8) and (1.9). This time we consider

$$\frac{\sin(n + \frac{1}{2})\theta}{(n + \frac{1}{2})} = \int_0^\theta \cos\left(n + \frac{1}{2}\right) \varphi \, d\varphi. \quad (1.15)$$

This is equivalent to

$$2 \frac{P_n^{(\frac{1}{2}, -\frac{1}{2})}(\cos \theta)}{P_n^{(\frac{1}{2}, -\frac{1}{2})}(1)} \sin \frac{\theta}{2} = \int_0^\theta \frac{P_n^{(-\frac{1}{2}, \frac{1}{2})}(\cos \varphi)}{P_n^{(-\frac{1}{2}, \frac{1}{2})}(1)} \cos \frac{\varphi}{2} \, d\varphi. \quad (1.16)$$

If we set $x = \cos \theta$ and $y = \cos \varphi$, (1.16) becomes

$$2 \frac{P_n^{(\frac{1}{2}, -\frac{1}{2})}(x)}{P_n^{(\frac{1}{2}, -\frac{1}{2})}(1)} (1-x)^{\frac{1}{2}} = \int_x^1 \frac{P_n^{(-\frac{1}{2}, \frac{1}{2})}(y)}{P_n^{(-\frac{1}{2}, \frac{1}{2})}(1)} \frac{dy}{(1-y)^{\frac{1}{2}}}. \quad (1.17)$$

Here we have P_n on both sides of the equation and because of this we can find a very general result for Jacobi polynomials which contains (1.17). This result was found by Bateman [7] and is fairly well known.

$$(1-x)^{\alpha+\mu} \frac{P_n^{(\alpha+\mu, \beta-\mu)}(x)}{P_n^{(\alpha+\mu, \beta-\mu)}(1)} = \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\Gamma(\mu)} \int_x^1 (1-y)^\alpha \frac{P_n^{(\alpha, \beta)}(y)}{P_n^{(\alpha, \beta)}(1)} (y-x)^{\mu-1} dy. \quad (1.18)$$

We could continue giving more results for trigonometric functions and then state the corresponding formulas for Jacobi polynomials. We will content ourselves with one more. It is easy to check that

$$\frac{\sin n\varphi}{n} = \int_\varphi^\pi \left(\frac{\sin \frac{\varphi}{2}}{\sin \frac{\theta}{2}} \right)^{2n} \frac{\sin(2n-1)\frac{\theta}{2}}{\sin \frac{\theta}{2}} d\theta. \quad (1.19)$$

The proof follows from

$$\frac{d}{dy} \frac{\sin \alpha y}{\alpha (\sin y)^\alpha} = - \frac{\sin(\alpha-1)y}{(\sin y)^{\alpha+1}} \quad (1.20)$$

which is a simple exercise.

Using (1.7) and (1.8) we see that (1.19) becomes

$$P_{n-1}^{(\frac{1}{2}, \frac{1}{2})}(\cos \varphi) \sin \varphi = (2n-1) \int_{\varphi}^{\pi} \left(\frac{\sin \frac{\varphi}{2}}{\sin \frac{\theta}{2}} \right)^{2n} P_{n-1}^{(\frac{1}{2}, \frac{1}{2})}(\cos \theta) d\theta. \quad (1.21)$$

Again a change of variables gives

$$\frac{P_n^{(\frac{1}{2}, -\frac{1}{2})}(x) (1-x^2)^{\frac{1}{2}}}{(1-x)^{n+1}} = (2n+1) \int_{-1}^x \frac{P_n^{(\frac{1}{2}, -\frac{1}{2})}(y) dy}{(1-y)^{n+1} (1-y^2)^{\frac{1}{2}}}. \quad (1.22)$$

There is a generalization of (1.22) which we will obtain later. It is

$$\begin{aligned} & \frac{(1+x)^{\beta+\mu}}{(1-x)^{n+\beta+1}} P_n^{(\alpha, \beta+\mu)}(x) \\ &= \frac{2^\mu \Gamma(n+\beta+\mu+1)}{\Gamma(n+\beta+1) \Gamma(\mu)} \int_{-1}^x \frac{(1+y)^\beta P_n^{(\alpha, \beta)}(y)}{(1-y)^{n+\beta+\mu+1}} (x-y)^{\mu-1} dy. \end{aligned} \quad (1.23)$$

As we will show later (1.23) is a generalization of an important integral which was found independently by Feldheim [18] and Vilenkin [32]. [For aesthetic reasons (1.23) is a nicer formula than the Feldheim-Vilenkin formula as we will see. For the sake of honesty we should mention that (1.23) was found first and (1.19) was discovered as a special case of it. We would also like to thank Professor A. Erdelyi for pointing out that the first formula like (1.23) that one of us "proved" could not possibly be correct. Also a conversation with Dr. T. P. Higgins was very useful.]

We conclude this introduction with some applications which can be obtained from these and other representations which we obtain.

In [18] Feldheim proved that

$$\sum_{k=0}^n \frac{P_k^{(\alpha, \beta)}(x)}{P_k^{(\alpha, \alpha)}(1)} \geq 0, \quad -1 \leq x \leq 1, \quad n = 0, 1, \dots \quad (1.24)$$

for $\alpha = \beta \geq 0$ and remarked that it also holds for some Jacobi polynomials. Feldheim knew Bateman's integral and so he must have known (1.24) for $\alpha \geq |\beta|$. He probably also knew (1.24) for $\alpha \geq -\beta - \frac{1}{2}$ when $-1 < \beta \leq -\frac{1}{2}$. We will show that (1.24) also holds for $\alpha \geq 0$ for $-\frac{1}{2} < \beta < 0$ and for a slightly larger region than $\alpha \geq -\beta - \frac{1}{2}$ for $-1 < \beta < -\frac{1}{2}$.

(1.19) can be used to give an easy proof of the following theorem of Turán [31]. See [4].

THEOREM A. If $\sum_{n=1}^N a_n \sin(2n-1)x \geq 0$, $0 < x < \pi$, and not all $a_n = 0$, then $\sum_{n=1}^N a_n/n \sin nx > 0$, $0 < x < \pi$.

The most important case of this theorem is $a_n = 1$. The proof we obtain in this way of Jackson's theorem

$$\sum_{n=1}^N \frac{\sin nx}{n} > 0, \quad 0 < x < \pi \quad (1.25)$$

compares favorably with Landau's inductive proof [23], which has been the easiest proof up to this time.

Finally as one more application let us mention the following generalization of a theorem of Seidel and Szász [27].

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n \prod_{j=1}^k \frac{P_j^{(\alpha_j, \beta_j)}(x_j)}{P_j^{(\alpha_j, \beta_j)}(1)} \cos n\theta > 0, \quad -1 < r < 1, \quad 0 \leq \theta \leq \pi, \quad (1.26)$$

for $\alpha_j \geq \max(\beta_j, -1 - \beta_j)$ Seidel and Szász prove (1.26) for $\alpha = \beta$ and $k = 1$.

2. HYPERGEOMETRIC FUNCTIONS

We have been unable to find a proof of (1.23) which uses just Jacobi polynomials. For this reason we must say something about hypergeometric functions.

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k. \quad (2.1)$$

See [5], [10] or [28] for a general discussion of ${}_2F_1$'s and of more general series of this type. The Jacobi polynomial $P_n^{(\alpha, \beta)}(x)$ can be written as a ${}_2F_1$.

$$\frac{P_n^{(\alpha, \beta)}(x)}{P_n^{(\alpha, \beta)}(1)} = {}_2F_1\left(-n, n + \alpha + \beta + 1; \alpha + 1; \frac{1-x}{2}\right). \quad (2.2)$$

By $(-n)_k$ we mean $(-n)(-n+1)\cdots(-n+k-1)$, which agrees with our definition of $(\alpha)_k$ in (1.4) if we let $\alpha \rightarrow -n$. Observe that

$$(1-x)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} x^n = F(a, b; b; x). \quad (2.3)$$

We will use the usual convention of dropping the subscripts on ${}_2F_1$.

The most important properties of hypergeometric functions that we use all follow from the following integral.

THEOREM 2.1. *If $\mu > 0$, $c > 0$, and $-1 < x < 1$ then*

$$F(a, b; c + \mu; x) = \frac{\Gamma(c + \mu)}{\Gamma(c) \Gamma(\mu)} \int_0^1 y^{c-1} (1-y)^{\mu-1} F(a, b; c; xy) dy. \quad (2.4)$$

This is an integral which was found by Bateman [7] and has since been rediscovered by a number of people. The proof is trivial. Just expand $F(a, b; c; y)$ in the series (2.1) and use Euler's formula for the beta function

$$\int_0^1 t^{a-1} (1-t)^{b-1} dt = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}. \quad (2.5)$$

If we let $c = b$ in (2.4) and use (2.3) we obtain Euler's integral

$$F(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tx)^{-a} dt, \quad (2.6)$$

where $b > 0$, $c > b$.

From (2.6) and a simple change of variables, $1-t=s$, we obtain

$$F(a, b; c; x) = (1-x)^{-a} F\left(a, c-b; c; \frac{x}{x-1}\right),$$

$$|x| < 1, \quad \left| \frac{x}{x-1} \right| < 1. \quad (2.7)$$

If we use $F(a, b; c; x) = F(b, a; c; x)$ we then get

$$F(a, b; c; x) = (1-x)^{-b} F\left(c-a, b; c; \frac{x}{x-1}\right),$$

$$|x| < 1, \quad \left| \frac{x}{x-1} \right| < 1. \quad (2.8)$$

From (2.7) and (2.8) we have

$$F(a, b; c; x) = (1-x)^{c-a-b} F(c-a, c-b; c; x), \quad |x| < 1. \quad (2.9)$$

The above proofs are due to Kummer [22] and are much easier than the proofs given in most of the current books on hypergeometric functions. They do require some conditions on the parameters which are not necessary for the final formulas, but a simple analytic continuation takes care of that. We have included this derivation for the sake of the average reader; who, like us, has

had very little contact with hypergeometric functions. Of the three books we referred to above only [10] even gives Bateman's integral and the derivation of (2.8) and (2.9) given there, which is due to Jacobi, is much more complicated than Kummer's earlier proof.

We will need two other integrals connecting hypergeometric functions with different parameters.

THEOREM 2.2. *If $\mu > 0$, $b > \mu$ and $-1 < y < 1$ then*

$$F(a, b - \mu; c; y) = \frac{\Gamma(b)}{\Gamma(\mu)\Gamma(b - \mu)} \int_0^1 x^{b-\mu-1} (1-x)^{\mu-1} F(a, b; c; xy) dx. \quad (2.10)$$

This can be proved in the same way that Theorem 2.1 was proved, i.e. expand and integrate term by term.

The other is a more complicated formula.

THEOREM 2.3. *If $\mu > 0$, $c > 0$, and $0 < y < 1$, then*

$$\begin{aligned} & y^{c+\mu-1} (1-y)^{a-c} F(a, b + \mu; c + \mu; y) \\ &= \frac{\Gamma(c + \mu)}{\Gamma(c) \Gamma(\mu)} \int_0^y (y-x)^{\mu-1} (1-x)^{a-c-\mu} x^{c-1} F(a, b; c; x) dx. \end{aligned} \quad (2.11)$$

We can no longer just integrate term by term because we don't have a simple enough expression for $\int_0^y (y-x)^l (1-x)^m x^n dx$. However we can use (2.4) and (2.7) to derive (2.11). Using (2.7) in (2.4) we obtain

$$\begin{aligned} & (1-x)^{-a} F\left(a, c + \mu - b; c + \mu; \frac{x}{x-1}\right) \\ &= \frac{\Gamma(c + \mu)}{\Gamma(c) \Gamma(\mu)} \int_0^1 y^{c-1} (1-y)^{\mu-1} (1-xy)^{-a} F\left(a, c - b; c; \frac{xy}{xy-1}\right) dy. \end{aligned} \quad (2.12)$$

Letting $t = x/(x-1)$ and $s = (xy)/(xy-1)$ and replacing $c-b$ by b we have (2.11).

The same type of reasoning can be used on Theorem 2.2 to obtain a fractional integral for $F(a, b + \mu; c; y)$ in terms of $F(a, b; c; x)$. But this integral does not lead to anything we can use for Jacobi polynomials so we will leave its statement to the reader who may have different interests from ours.

We should say something about the history of these results. Except for (2.11) they are all known and (2.11) was probably known to a number of people. We haven't found it in the literature. The others have been found often and rather than give some of the references and miss others we only

gave Bateman's original paper, which seems to have been the first. One reason these results have been forgotten and then rediscovered is that most of the people who proved them didn't give any applications of their results. This seems to be the reason that these results have not been included in books on hypergeometric functions. We give one application now of Theorem 2.3 to show that these theorems can be used to obtain interesting results. If we let $\mu = 1$ we have

$$y^c(1-y)^{a-c}F(a, b+1; c+1; y) = c \int_0^y (1-x)^{a-c-1} x^{c-1} F(a, b; c; x) dx. \quad (2.13)$$

We can differentiate (2.13) and obtain

$$\frac{d}{dy} y^c(1-y)^{a-c}F(a, b+1; c+1; y) = c(1-y)^{a-c-1} y^{c-1} F(a, b; c; y). \quad (2.14)$$

(2.14) was found by Bailey [6]. The above proof, including the proof of Theorem 2.3, is easier than his proof.

3. JACOBI POLYNOMIALS

We now specialize the results of the previous section to Jacobi polynomials. Recall that

$$\frac{P_n^{(\alpha, \beta)}(x)}{P_n^{(\alpha, \beta)}(1)} = F\left(-n, n + \alpha + \beta + 1; \alpha + 1; \frac{1-x}{2}\right). \quad (3.1)$$

For $\alpha, \beta > -1$, $P_n^{(\alpha, \beta)}(x)$ form an orthogonal set of functions on $(-1, 1)$ with respect to the measure $(1-x)^\alpha (1+x)^\beta dx$. We make the restriction that α and β satisfy $\alpha, \beta > -1$ in all that follows and will not repeat it. $P_n^{(\alpha, \beta)}(x)$ satisfies the following type of symmetry condition which we will find useful.

$$P_n^{(\alpha, \beta)}(-x) = (-1)^n P_n^{(\beta, \alpha)}(x). \quad (3.2)$$

If we use (3.1) in (2.4) we obtain the following:

$$(1-x)^{\alpha+\mu} \frac{P_n^{(\alpha+\mu, \beta-\mu)}(x)}{P_n^{(\alpha+\mu, \beta-\mu)}(1)} = \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\Gamma(\mu)} \int_x^1 (1-y)^\alpha \frac{P_n^{(\alpha, \beta)}(y)}{P_n^{(\alpha, \beta)}(1)} (y-x)^{\mu-1} dy. \quad (3.3)$$

In (3.3) and in what follows we will have $\mu > 0$ and $-1 < x, y < 1$. If we use (3.2) we have

$$\begin{aligned} & (1+x)^{\beta+\mu} \frac{P_n^{(\alpha-\mu, \beta+\mu)}(x)}{P_n^{(\alpha-\mu, \beta+\mu)}(-1)} \\ &= \frac{\Gamma(\beta+\mu+1)}{\Gamma(\beta+1)\Gamma(\mu)} \int_{-1}^x (1+y)^\beta \frac{P_n^{(\alpha, \beta)}(y)}{P_n^{(\alpha, \beta)}(-1)} (x-y)^{\mu-1} dy. \end{aligned} \quad (3.4)$$

From (2.10) we have

$$\begin{aligned} & (1+x)^{n+\alpha+\beta} P_n^{(\alpha-\mu, \beta)}(x) \\ &= \frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1-\mu)\Gamma(\mu)} \int_{-1}^x (1+y)^{n+\alpha+\beta-\mu} P_n^{(\alpha, \beta)}(y) (x-y)^{\mu-1} dy. \end{aligned} \quad (3.5)$$

Again using (3.2) we have

$$\begin{aligned} & (1-x)^{n+\alpha+\beta} P_n^{(\alpha, \beta-\mu)}(x) \\ &= \frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1-\mu)\Gamma(\mu)} \int_x^1 (1-y)^{n+\alpha+\beta-\mu} P_n^{(\alpha, \beta)}(y) (y-x)^{\mu-1} dy. \end{aligned} \quad (3.6)$$

In (3.5) and (3.6) if $n=0$ we must assume $\alpha+\beta-\mu > -1$, instead of just $\alpha-\mu > -1$ or $\beta-\mu > -1$. Finally, using (2.11) first and then (3.2) we have

$$\begin{aligned} & \frac{(1-x)^{\alpha+\mu}}{(1+x)^{n+\alpha+1}} \frac{P_n^{(\alpha+\mu, \beta)}(x)}{P_n^{(\alpha+\mu, \beta)}(1)} \\ &= \frac{2^\mu \Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\Gamma(\mu)} \int_x^1 \frac{(1-y)^\alpha}{(1+y)^{n+\alpha+\mu+1}} \frac{P_n^{(\alpha, \beta)}(y)}{P_n^{(\alpha, \beta)}(1)} (y-x)^{\mu-1} dy \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} & \frac{(1+x)^{\beta+\mu}}{(1-x)^{n+\beta+1}} \frac{P_n^{(\alpha, \beta+\mu)}(x)}{P_n^{(\alpha, \beta+\mu)}(-1)} \\ &= \frac{2^\mu \Gamma(\beta+\mu+1)}{\Gamma(\beta+1)\Gamma(\mu)} \int_{-1}^x \frac{(1+y)^\beta}{(1-y)^{n+\beta+\mu+1}} \frac{P_n^{(\alpha, \beta)}(y)}{P_n^{(\alpha, \beta)}(-1)} (x-y)^{\mu-1} dy. \end{aligned} \quad (3.8)$$

(3.5) and (3.7) include important results for ultraspherical polynomials in the cases $\beta = \pm \frac{1}{2}$. Ultraspherical polynomials are Jacobi polynomials with equal parameters α and β . However, for historical and some practical reasons, they are normalized differently. We define $C_n^\lambda(x)$ by

$$C_n^\lambda(x) = \frac{(2\lambda)_n}{(\lambda + \frac{1}{2})_n} P_n^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})}(x). \quad (3.9)$$

There is another connection between ultraspherical and Jacobi polynomials.

$$\frac{P_{2n}^{(\alpha, \alpha)}(x)}{P_{2n}^{(\alpha, \alpha)}(1)} = \frac{P_n^{(\alpha, -\frac{1}{2})}(2x^2 - 1)}{P_n^{(\alpha, -\frac{1}{2})}(1)} = \frac{C_{2n}^{\alpha + \frac{1}{2}}(x)}{C_{2n}^{\alpha + \frac{1}{2}}(1)} \quad (3.10)$$

$$\frac{P_{2n+1}^{(\alpha, \alpha)}(x)}{P_{2n+1}^{(\alpha, \alpha)}(1)} = \frac{x P_n^{(\alpha, \frac{1}{2})}(2x^2 - 1)}{P_n^{(\alpha, \frac{1}{2})}(1)} = \frac{C_{2n+1}^{\alpha + \frac{1}{2}}(x)}{C_{2n+1}^{\alpha + \frac{1}{2}}(1)}. \quad (3.11)$$

If we choose $\beta = \pm \frac{1}{2}$ in (3.5) and (3.7) and use (3.10) and (3.11) and then do some tedious calculations we see that the following results hold.

$$C_n^\lambda(x) = \frac{2\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu - \lambda)} \int_0^1 t^{n+2\lambda-1} C_n^\nu(xt) (1 - t^2)^{\nu-\lambda-1} dt, \quad \nu > \lambda. \quad (3.12)$$

Again, if $n = 0$ we must have $\lambda > 0$ instead of $\lambda > -\frac{1}{2}$.

$$\begin{aligned} \frac{C_n^\nu(\cos \theta) \sin^{2\nu-1} \theta}{C_n^\nu(1) \cos^{n+2\lambda+1} \theta} &= \frac{2\Gamma(\nu + \frac{1}{2})}{\Gamma(\lambda + \frac{1}{2})\Gamma(\nu - \lambda)} \int_0^\theta \frac{\sin^{2\lambda} \psi [\cos^2 \psi - \cos^2 \theta]^{\nu-\lambda-1}}{\cos^{n+2\nu} \psi} \\ &\quad \cdot \frac{C_n^\lambda(\cos \psi)}{C_n^\lambda(1)} d\psi, \quad 0 < \theta < \frac{\pi}{2}, \quad \nu > \lambda. \end{aligned} \quad (3.13)$$

$0 < \theta < \pi/2, \nu > \lambda$.

(3.13) is a result found by Feldheim [18] and Vilenkin [32] and as we will see it has a number of interesting applications. In fact the surprising thing is that while (3.13) is very useful, (3.7) and (3.8) which contain (3.13) are not as useful. The easier result (3.3) is more useful for most problems as we shall see.

The proof we gave of (3.13), i.e. proving (3.7) first, is different from the proofs given by Feldheim and Vilenkin. Feldheim uses Sonine's first integral for Bessel functions and Vilenkin uses a method which was used by Watson [34] to give a proof of Sonine's integral. Neither of these proofs seems to generalize to Jacobi polynomials. For variety we will give one more proof of (3.7), this time by Laplace transforms. Before we give it we should make some remarks about the difference between (3.7) and (3.13). The kernel which occurs in (3.7) is a simpler function than the one which occurs in (3.13).

Also (3.7) holds for all values of the variable x , $-1 < x < 1$, while (3.13) only holds for $0 < \theta < \pi/2$ (or $0 < x < 1$), and a different formula holds for $\pi/2 < \theta < \pi$. The reason for this difference seems to be tied up with the monotone behavior of the zeros of Jacobi polynomials $P_n^{(\alpha+\mu, \beta)}(x)$ as μ increases and the fact that the zeros of $P_n^{(\alpha+\mu, \alpha+\mu)}(x)$ tend monotonically to zero and not one of the end points. Both Feldheim and Vilenkin obtain the following result which is equivalent to (3.13). It holds for all values of the variable but now the argument of C_n^λ is so complicated that one is tempted to look for another formula. (3.7) is this other formula.

$$\frac{C_n^\nu(\cos \theta)}{C_n^\nu(1)} = \frac{2\Gamma(\nu + \frac{1}{2})}{\Gamma(\lambda + \frac{1}{2})\Gamma(\nu - \lambda)} \int_0^{\pi/2} \sin^{2\lambda} \varphi \cos^{2\nu-2\lambda-1} (1 - \sin^2 \theta \cos^2 \varphi)^{n/2} \\ \cdot \frac{C_n^\lambda(\cos \theta (1 - \sin^2 \theta \cos^2 \varphi)^{-\frac{1}{2}})}{C_n^\lambda(1)} d\varphi, \quad \nu > \lambda, \quad 0 \leq \theta \leq \pi. \quad (3.14)$$

In addition to (1.19) and (1.20) we should mention a few other special cases of our results. These can be checked directly and some of them are even useful.

$$\frac{d}{d\theta} \sin^n \theta \cos n\theta = n \sin^{n-1} \theta \cos(n+1)\theta \quad (3.15)$$

$$\frac{d}{d\theta} \frac{\sin n\theta}{n(\cos \theta)^n} = \frac{\cos(n-1)\theta}{(\cos \theta)^{n+1}} \quad (3.16)$$

$$\frac{d}{d\theta} \cos^n \theta \cos n\theta = -n \cos^{n+1} \theta \sin(n+1)\theta. \quad (3.17)$$

Once you see the pattern of results of this type they are easy to write down. These expressions are probably not new but they are far from well known and they can be extremely useful. As we remarked earlier, (1.19) can be used to prove an interesting theorem of P. Turán. Later, when we give some generalizations of the following inequality of Lyness and Moler [24] we will show how (3.16) gives a very simple proof of their inequality.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{\sin n\theta}{n} \right)^{2k} \geq 0 \quad \text{for} \quad k = 1, 2, \dots \quad (3.18)$$

We now return to (3.8) and give a different proof of it. There is one thing wrong aesthetically with (3.8) and that is the occurrence of μ in the term $(1-y)^{-n-\beta-\mu-1}$ on the right hand side. Ideally what the formula should be is a given function $g_\beta(y)$ on the right which depends on β and on the left we should have $g_{\beta+\mu}(x)$. If we can achieve this then we should be able to prove (3.8) by taking Laplace transforms. We set

$$y = \frac{r-1}{r+1}, \quad x = \frac{s-1}{s+1}.$$

Then (3.8) becomes

$$\frac{s^{\beta+\mu}(1+s)^n P_n^{(\alpha, \beta+\mu)}\left(\frac{s-1}{s+1}\right)}{\Gamma(n+\beta+\mu+1)} = \frac{1}{\Gamma(\mu)} \int_0^s \frac{r^\beta(1+r)^n P_n^{(\alpha, \beta)}\left(\frac{r-1}{r+1}\right) (s-r)^{\mu-1} dr}{\Gamma(n+\beta+1)} \quad (3.19)$$

$s > 0$. If we set

$$g_\beta(s) = \frac{s^\beta(1+s)^n P_n^{(\alpha, \beta)}\left(\frac{s-1}{s+1}\right)}{\Gamma(n+\beta+1)} \quad (3.20)$$

then (3.19) can be written

$$g_{\beta+\mu}(s) = \frac{1}{\Gamma(\mu)} \int_0^s (s-r)^{\mu-1} g_\beta(r) dr. \quad (3.21)$$

Taking Laplace transforms we see that to prove (3.21) it is sufficient to show that $\int_0^\infty g_\beta(s) e^{-xs} ds$ is a function of the form $A(x)/x^\beta$ where $A(x)$ is independent of β . This is so because

$$\frac{1}{\Gamma(\mu)} \int_0^\infty s^{\mu-1} e^{-xs} ds = x^{-\mu}. \quad (3.22)$$

and then for the Laplace transform of (3.21) we would have

$$\frac{A(x)}{x^{\beta+\mu}} = \frac{1}{x^\mu} \cdot \frac{A(x)}{x^\beta}, \quad (3.23)$$

which is trivially true. Then the uniqueness theorem for Laplace transforms would give (3.19), which is equivalent to (3.8). But we know exactly what

$$\int_0^\infty \frac{s^\beta(1+s)^n P_n^{(\alpha, \beta)}\left(\frac{s-1}{s+1}\right)}{\Gamma(n+\beta+1)} e^{-xs} ds \quad (3.24)$$

is. Since

$$P_n^{(\alpha, \beta)}(y) = \binom{n+\beta}{n} \left(\frac{y-1}{2}\right)^n F\left(-n, -n-\alpha; \beta+1; \frac{y+1}{y-1}\right)$$

a simple computation shows that (3.24) is

$$\frac{(-2)^n}{x^{\beta+1}n!} \sum_{k=0}^n \frac{(-1)^k (-n)_k (-n-\alpha)_k}{k!} x^{-k} \quad (3.25)$$

which is what we wanted to show, since the only way β enters into this formula is $x^{-\beta}$. However we can simplify (3.25) and obtain an integral which connects Jacobi and Laguerre polynomials. If we let $j = n - k$ and sum on j we see (3.25) is

$$\frac{(-n-\alpha)_n(-n)_n}{x^{n+\beta+1}(1)_n(1)_n} \sum_{j=0}^n \frac{(-n)_j x^j}{(1+\alpha)_j j!}. \quad (3.26)$$

Recall that the Laguerre polynomial $L_n^\alpha(s)$ is defined by

$$L_n^\alpha(x) = \frac{(1+\alpha)_n}{n!} {}_1F_1(-n; x), \quad (3.27)$$

so that (3.26) is really

$$\frac{(-n-\alpha)_n(-n)_n}{x^{n+\beta+1}(1)_n(1)_n(1+\alpha)_n} n! L_n^\alpha(x). \quad (3.28)$$

We can simplify (3.28) and obtain

$$\int_0^\infty \frac{s^\beta (1+s)^n P_n^{(\alpha, \beta)}\left(\frac{s-1}{s+1}\right) e^{-xs}}{\Gamma(n+\beta+1)} ds = \frac{L_n^\alpha(x)}{x^{n+\beta+1}}. \quad (3.29)$$

(3.29) is not new. A similar formula was obtained by J. Chaudhuri [9] and it is undoubtedly contained in one of the formulas in ([12], Section 4.21). One interesting fact is that (3.29) is a consequence of (3.8). There are some other integrals involving Laguerre polynomials and Jacobi polynomials which we will now list. Then we will show how they can all be obtained from the integrals we have for Jacobi polynomials.

The only well known integral is

$$\frac{x^{\alpha+\mu} L_n^{\alpha+\mu}(x)}{\Gamma(n+\alpha+\mu+1)} = \frac{1}{\Gamma(\mu)} \int_0^x \frac{(x-y)^{\mu-1} y^\alpha L_n^\alpha(y) dy}{\Gamma(n+\alpha+1)}, \quad \mu > 0. \quad (3.30)$$

This is due to Koshliakov [21]. There is an inverse to (3.30) which isn't as well known.

$$e^{-x} L_n^\alpha(x) = \frac{1}{\Gamma(\mu)} \int_x^\infty (y-x)^{\mu-1} e^{-y} L_n^{\alpha+\mu}(y) dy, \quad \mu > 0. \quad (3.31)$$

A direct proof form (3.30) is found in [3].

Another formula is

$$\Gamma(n+\alpha+\beta+1) P_n^{(\alpha, \beta)}(x) = \int_0^\infty u^{n+\alpha+\beta} e^{-u} L_n^\alpha\left[\frac{u(1-x)}{2}\right] du. \quad (3.32)$$

This was found by Feldheim [15].

(3.30) and (3.31) are ordinary fractional integrals and (3.29) and (3.32) are Laplace transforms, or fractional integrals of infinite order if you want to think of them in this way. It is a fruitful way, as it leads one to see where gaps are in the literature and sometimes even leads to proofs.

These four formulas all follow easily from

$$L_n^\alpha(x) = \lim_{\beta \rightarrow \infty} P_n^{(\alpha, \beta)} \left(1 - \frac{2x}{\beta} \right). \quad (3.33)$$

and the formulas we derived at the beginning of this chapter. We omit the details.

Similarly it is possible to derive the Sonine integrals for Bessel functions from the above results. Here we use

$$\lim_{n \rightarrow \infty} n^{-\alpha} L_n^\alpha \left(\frac{x}{n} \right) = x^{-\alpha/2} J_\alpha(2x^{1/2}) \quad (3.34)$$

and

$$\lim_{n \rightarrow \infty} n^{-\alpha} P_n^{(\alpha, \beta)} \left(\cos \frac{\theta}{n} \right) = \left(\frac{\theta}{2} \right)^{-\alpha} J_\alpha(\theta). \quad (3.35)$$

The first Sonine integral is

$$z^{\mu+\nu+1} J_{\mu+\nu+1}(z) = \frac{1}{2^\mu \Gamma(\mu+1)} \int_0^z r^{\nu+1} J_\nu(r) (z^2 - r^2)^\mu dr. \quad (3.36)$$

Another integral found by Sonine which is extremely useful and is not given in most of the standard references is

$$z^{-\nu} J_\nu(z) = \frac{2^{\nu-\mu+1}}{\Gamma(\nu-\mu)} \int_z^\infty y^{-\mu+1} J_\mu(y) (y^2 - z^2)^{\mu-\nu-1} dy, \quad (3.37)$$

$$-1 < \nu < \mu < 2\nu + \frac{3}{2}. \quad (3.37)$$

This is the inverse of (3.36) under Hankel transforms, just as (3.31) is an inverse of (3.30). There is a generalization of (3.36) which is usually known as Sonine's second integral.

$$\begin{aligned} & J_{\mu+\nu+1}[(a^2 + b^2)^{1/2}] a^\mu b^\nu (a^2 + b^2)^{-(\mu+\nu+1)/2} \\ &= \int_0^{\pi/2} J_\mu(a \sin \theta) J_\nu(b \cos \theta) (\sin \theta)^{\mu+1} (\cos \theta)^{\nu+1} d\theta. \end{aligned} \quad (3.38)$$

There are two formulas for the classical orthogonal polynomials which contain (3.38), and they are both formulas for Laguerre polynomials.

One is

$$\frac{L_{m+n}^{\alpha}(x)}{L_{m+n}^{\alpha}(0)} = \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha-\gamma)} \int_0^1 t^{\gamma}(1-t)^{\alpha-\gamma} \frac{L_m^{\gamma}(xt)}{L_m^{\gamma}(0)} \frac{L_n^{\alpha-\gamma-1}[x(1-t)]}{L_n^{\alpha-\gamma-1}(0)} dt. \quad (3.39)$$

This is due to Feldheim [17]. Feldheim also gives a reference to a paper of Tricomi which implicitly contains (3.39).

The second formula is a sum rather than an integral and we will give it later. It would be very useful to have analogues of (3.38) for Jacobi or ultraspherical polynomials.

There has been interest in some mathematical circles to explain properties of special functions from an algebraic point of view, either group representations or from Lie algebra. The Sonine integrals were derived by Orihara [26] and many other results were derived in Vilenkin [33] and Miller [25]. The machinery that is needed by these approaches is very formidable and we suggest that an analogue of Sonine's second integral for Jacobi polynomials would be an interesting test problem, to see if the machinery that has been developed there is powerful enough to find some new results for the classical functions.

Finally we list dual expansions to those we have derived. In the above we held n fixed and integrated on x to obtain our formula. Now we hold x fixed and sum on n .

$$\begin{aligned} P_n^{(\gamma, \beta)}(x) &= \frac{\Gamma(n+\beta+1)}{\Gamma(\gamma-\alpha)\Gamma(n+\beta+\gamma+1)} \\ &\times \sum_{k=0}^n \frac{\Gamma(n-k+\gamma-\alpha)\Gamma(n+k+\beta+\gamma+1)}{\Gamma(n-k+1)\Gamma(n+k+\alpha+\beta+2)} \\ &\times \frac{\Gamma(k+\alpha+\beta+1)(2k+\alpha+\beta+1)}{\Gamma(k+\beta+1)} P_k^{(\alpha, \beta)}(x). \end{aligned} \quad (3.40)$$

This is (9.4.1) in [30]. Its inverse is

$$\begin{aligned} &(1-x)^{\alpha} P_n^{(\alpha, \beta)}(x) \\ &= \sum_{k=n}^{\infty} \frac{\Gamma(n+\alpha+1)\Gamma(n+k+\beta+\gamma+1)\Gamma(k-n+\gamma-\alpha)}{\Gamma(n+1)\Gamma(n+k+\alpha+\beta+2)\Gamma(k-n+1)} \\ &\times \frac{\Gamma(k+1)(2k+\gamma+\beta+1)}{\Gamma(k+\gamma+1)\Gamma(\gamma-\alpha)2^{\gamma-\alpha}} P_k^{(\gamma, \beta)}(x) (1-x)^{\gamma}, \end{aligned} \quad (3.41)$$

where we need $\alpha > \gamma/2 - \frac{3}{4}$ to have convergence of the series. This is given in [3].

For $\beta = \pm \frac{1}{2}$, (3.40) and (3.41) contain two very useful formulas for ultraspherical polynomials. We use (3.10) and (3.11) in (3.40) and (3.41) and obtain

$$C_n^\lambda(x) = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\lambda-\nu)} \times \sum_{k=0}^{[n/2]} \frac{(n-2k+\nu)\Gamma(k+\lambda-\nu)\Gamma(n-k+\lambda)}{k!\Gamma(n-k+\nu+1)} C_{n-2k}^\nu(x) \quad (3.42)$$

$$(1-x^2)^{\lambda-\frac{1}{2}} C_n^\lambda(x) = \sum_{k=0}^{\infty} \alpha_{k,n}^{\lambda,\mu} C_{n+2k}^\mu(x) (1-x^2)^{\mu-\frac{1}{2}} \quad (3.43)$$

where

$$\alpha_{k,n}^{\lambda,\mu} = \frac{\left(\frac{\Gamma(\mu) 2^{2\mu-2\lambda} (n+2k+\mu)(n+2k)!}{\Gamma(\mu-\lambda)\Gamma(\lambda)n!k!\Gamma(n+k+\lambda+1)\Gamma(n+2k+2\mu)} \times \Gamma(n+2\lambda)\Gamma(n+k+\mu)\Gamma(k+\mu-\lambda) \right)}{\lambda > \frac{\mu-1}{2}}.$$

(3.42) was found by Gegenbauer [20]. It contains the following well known formula.

$$C_n^\lambda(\cos \theta) = \sum_{k=0}^n \frac{(\lambda)_k (\lambda)_{n-k}}{k!(n-k)!} \cos(n-2k)\theta. \quad (3.44)$$

This follows from (3.42) if we let $\nu \rightarrow 0$.

Similarly (3.43) contains a formula of Szegő ([30], (4.9.22)).

$$(\sin \theta)^{2\lambda-1} C_n^\lambda(\cos \theta) = \frac{2^{1-2\lambda} \Gamma(n+2\lambda)}{\Gamma(\lambda)\Gamma(n+\lambda+1)} \sum_{k=0}^{\infty} f_k^{(\lambda)} \sin(n+2k+1)\theta,$$

$$f_0^{(\lambda)} = 1, \quad f_k^{(\lambda)} = \frac{(1-\lambda)(2-\lambda)\cdots(k-\lambda)}{1 \cdot 2 \cdots k} \frac{(n+1)_k}{(n+\lambda+1)_k}, \quad \lambda > 0. \quad (3.45)$$

For Laguerre polynomials we have

$$L_n^{\alpha+\mu}(x) = \sum_{k=0}^n \frac{(\mu)_{n-k}}{(n-k)!} L_k^\alpha(x). \quad (3.46)$$

As an analogue of the Sonine second integral we have

$$L_n^{(\alpha+\beta+1)}(x+y) = \sum_{k=0}^n L_k^\alpha(x) L_{n-k}^\beta(y). \quad (3.47)$$

Finally we consider ultraspherical polynomials in a bit more detail. The real reason for the different normalization for $C_n^\lambda(x)$ is the following generating function.

$$(1 - 2xr + r^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(x) r^n. \quad (3.48)$$

A simple computation shows that

$$\frac{\lambda(1-r^2)}{(1-2xr+r^2)^{\lambda+1}} = \sum_{n=0}^{\infty} (n+\lambda) C_n^\lambda(x) r^n \geq 0 \quad \text{for } \lambda > 0. \quad (3.49)$$

When combined with the following generalized translation operator due to Bochner [8], (3.49) leads to the Poisson kernel for ultraspherical series.

As Bochner [8] pointed out, an old formula of Gegenbauer [19]

$$\frac{C_n^\lambda(x) C_n^\lambda(y)}{C_n^\lambda(1) C_n^\lambda(1)} = \frac{1}{c_\lambda} \int_0^\pi \frac{C_n^\lambda(xy + (1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}} \cos \theta)}{C_n^\lambda(1)} \sin^{2\lambda-1} \theta \, d\theta \quad (3.50)$$

where

$$c_\lambda = \int_0^\pi \sin^{2\lambda-1} \theta \, d\theta, \quad \lambda > 0, \quad (3.51)$$

leads to the following result.

If $f(x)$ has the expansion

$$f(x) \sim \sum_{n=0}^{\infty} a_n \frac{(n+\lambda)}{\lambda} C_n^\lambda(x), \quad (3.52)$$

where

$$a_n = \frac{1}{c_{\lambda+\frac{1}{2}}} \int_0^\pi f(x) \frac{C_n^\lambda(x)}{C_n^\lambda(1)} (1-x^2)^{\lambda-\frac{1}{2}} \, dx \quad (3.53)$$

then the function

$$f(x, y) = \frac{1}{c_\lambda} \int_0^\pi f(xy + (1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}} \cos \theta) \sin^{2\lambda-1} \theta \, d\theta$$

has the expansion

$$f(x, y) \sim \sum_{n=0}^{\infty} a_n \frac{(n+\lambda)}{\lambda} C_n^\lambda(x) \frac{C_n^\lambda(y)}{C_n^\lambda(1)}. \quad (3.54)$$

If $f(x) \geq 0$, $-1 \leq x \leq 1$, we have $f(x, y) \geq 0$, $-1 \leq x, y \leq 1$.

4. APPLICATIONS

We first consider a problem that Feldheim mentioned; when is

$$\sum_{k=0}^n \frac{P_k^{(\alpha, \beta)}(x)}{P_k^{(\alpha, \beta)}(1)} \geq 0? \quad (4.1)$$

Since

$$P_1^{(\alpha, \beta)}(x) = \left(\frac{\alpha + \beta + 2}{2} \right) x + \frac{\alpha - \beta}{2}, \quad P_0^{(\alpha, \beta)}(x) = 1, \quad (4.2)$$

we see that (4.1) fails for $\beta > \alpha$ if $n = 1$ and $x = -1$. Also for $n = 2$ and $\alpha = \beta = -\frac{1}{2}$ we have the sum

$$1 + \cos \theta + \cos 2\theta = \cos \theta (2 \cos \theta + 1) < 0 \quad (4.3)$$

for $\pi/2 < \theta < 2\pi/3$ as Szegő observed [18]. Thus (4.1) can not hold for all $\alpha = \beta$. Feldheim proved (4.1) for $\alpha = \beta \geq 0$ by the following method. First for $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$ we have more than (4.1). We have

$$\sum_{k=0}^n \frac{\sin(k + \frac{1}{2})\theta}{\sin \theta/2} = \sum_{k=0}^n \frac{\sin(2k+1)\theta/2}{\sin \theta/2} = \frac{1 - \cos(n+1)\theta}{2 \sin^2 \theta/2} \geq 0. \quad (4.4)$$

Now using Mehler's integral we have

$$\sum_{k=0}^n P_k(\cos \theta) = \frac{\sqrt{2}}{\pi} \int_0^\pi \frac{\sum_{k=0}^n \sin(k + \frac{1}{2})\varphi}{(\cos \theta - \cos \varphi)^{\frac{1}{2}}} d\varphi > 0. \quad (4.5)$$

Mehler's integral is just Bateman's integral (3.4) for $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$, $\mu = \frac{1}{2}$. This part of the argument is due to Fejér [13]. Then Feldheim observed that (3.14) implies

$$\sum_{k=0}^n \frac{P_k^{(\alpha, \alpha)}(\cos \theta)}{P_k^{(\alpha, \alpha)}(1)} \geq 0, \quad 0 \leq \theta \leq \pi, \quad \alpha > 0. \quad (4.6)$$

Since

$$P_k^{(\alpha, \alpha)}(-x) = (-1)^k P_k^{(\alpha, \alpha)}(x)$$

we have

$$\sum_{k=0}^n \frac{P_{2k}^{(\alpha, \alpha)}(\cos \theta)}{P_{2k}^{(\alpha, \alpha)}(1)} \geq 0, \quad 0 \leq \theta \leq \pi, \quad \alpha > 0. \quad (4.7)$$

Then using (3.10) we have

$$\sum_{k=0}^n \frac{P_k^{(\alpha, -\frac{1}{2})}(\cos 2\theta)}{P_k^{(\alpha, -\frac{1}{2})}(1)} \geq 0. \quad (4.8)$$

Applying the Bateman integral (3.3) to (4.6) and (4.8) we have (4.1) for $\alpha \geq \beta$, $\alpha \geq -\beta$ if $\beta \geq -\frac{1}{2}$ and $\alpha \geq -\beta - \frac{1}{2}$ if $-1 < \beta < -\frac{1}{2}$. Since Feldheim knew the Bateman integral he undoubtedly knew this result. Actually for $\beta = -\frac{1}{2}$ we have a stronger inequality than (4.1). For $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$ recall (4.4), which was stronger than (4.1). (4.4) can be written as

$$\sum_{k=0}^n (2k+1) \frac{P_k^{(\frac{1}{2}, -\frac{1}{2})}(\cos \theta)}{P_k^{(\frac{1}{2}, -\frac{1}{2})}(1)} \geq 0, \quad n = 0, 1, \dots \quad (4.9)$$

Now we apply (3.5) to (4.9) and obtain

$$\begin{aligned} & \sum_{k=0}^n \frac{\Gamma(k+1-\mu)(2k+1)}{\Gamma(k+1)P_k^{(\frac{1}{2}, -\frac{1}{2})}(1)} P_k^{(\frac{1}{2}-\mu, -\frac{1}{2})}(x) \\ &= \frac{1}{\Gamma(\mu)} \int_{-1}^x \sum_{k=0}^n \left(\frac{1+y}{1+x} \right)^k (2k+1) \frac{P_k^{(\frac{1}{2}, -\frac{1}{2})}(y)}{P_k^{(\frac{1}{2}, -\frac{1}{2})}(1)} (1+y)^{-\mu} (x-y)^{\mu-1} dy. \end{aligned} \quad (4.10)$$

Since $[(1+y)/(1+x)]^k$ is a decreasing function of k for $-1 \leq y \leq x$ we may sum by parts and use (4.9) to show that the right hand side is non-negative.

Now apply Bateman's integral (3.4) to (4.10). We have

$$\begin{aligned} & \sum_{k=0}^n \frac{P_k^{(\frac{1}{2}-\mu-\nu, -\frac{1}{2}+\nu)}(x)}{P_k^{(\frac{1}{2}-\mu-\nu, -\frac{1}{2}+\nu)}(1)} (1+x)^{\frac{1}{2}-\mu-\nu} \\ &= \frac{1}{\Gamma(\nu)} \int_{-1}^x \sum_{k=0}^n \frac{\Gamma(k+\frac{1}{2}+\nu) \Gamma(k+1) P_k^{(\frac{1}{2}-\mu, -\frac{1}{2})}(y)}{\Gamma(k+\frac{3}{2}-\mu-\nu) \Gamma(k+\frac{1}{2})} (1+y)^{\frac{1}{2}-\mu} (x-y)^{\nu-1} dy \\ &= \frac{1}{\Gamma(\nu)} \int_{-1}^x \sum_{k=0}^n \frac{\Gamma(k+\frac{1}{2}+\nu) \Gamma(k+1) \Gamma(k+1-\mu) P_k^{(\frac{1}{2}-\mu, -\frac{1}{2})}(y)}{\Gamma(k+\frac{3}{2}-\mu-\nu) \Gamma(k+1-\mu) \Gamma(k+\frac{1}{2})} \\ & \quad \times (1+y)^{\frac{1}{2}-\mu} (x-y)^{\nu-1} dy. \end{aligned} \quad (4.11)$$

If we can show that

$$\alpha_k = \frac{\Gamma(k + \frac{1}{2} + \nu) \Gamma(k + 1)}{\Gamma(k + \frac{3}{2} - \mu - \nu) \Gamma(k + 1 - \mu)}$$

is decreasing in k we can sum the right hand side of (4.11) by parts and show that the left hand side is nonnegative. A simple calculation show this is equivalent to

$$k[1 - 2\mu - 2\nu] + (\frac{3}{2} - \mu - \nu)(1 - \mu) - (\nu + \frac{1}{2}) \geq 0,$$

for $k = 0, 1, \dots$. If $\mu + \nu \leq \frac{1}{2}$ the first term is nonnegative and so is the second. Thus we have shown (4.1) holds for $\alpha \geq 0$ if $-\frac{1}{2} \leq \beta \leq 0$.

Another calculation shows that (4.10) is equivalent to

$$\sum_{k=0}^n C_{2k}^\lambda(x) \geq 0, \quad \frac{1}{2} \leq \lambda \leq 1. \quad (4.12)$$

For $\lambda < \frac{1}{2}$, (4.12) follows from (4.13)

$$\sum_{k=0}^n C_k^\lambda(x) \geq 0, \quad -\frac{1}{2} < \lambda \leq \frac{1}{2}, \quad \lambda \neq 0, \quad (4.13)$$

which was proven by Fejér [14] for $0 < \lambda \leq \frac{1}{2}$ and by Szegő [29] for $-\frac{1}{2} < \lambda < 0$. Another proof of (4.12) and (4.13) for $0 < \lambda \leq \frac{1}{2}$ is to apply (3.12) to the upper end point result, each of which we know. (4.12) can not be extended to larger values of λ . For $C_0^\lambda(x) = 1$ and $C_2^\lambda(x) = 2\lambda(1 + \lambda)x^2 - \lambda$ so $C_0^\lambda(x) + C_2^\lambda(x) = 1 - \lambda + 2\lambda(1 + \lambda)x^2$. For $x = 0$ this is negative if $\lambda > 1$.

A similar argument to that given above shows that (4.1) holds for $-1 < \beta \leq -\frac{1}{2}$ if α is to the right of the hyperbola

$$2\alpha^2 + 2\alpha\beta + 2\alpha + 2\beta + 1 = 0.$$

For $\beta = -1$, this hyperbola has $\alpha = (-1 + \sqrt{3})/2$, which is less than the $\frac{1}{2}$ we had previously.

We now consider the following result of Seidel and Szász [27] and various generalizations of it.

Let $\lambda > 0$. Then

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n \frac{C_n^\lambda(x)}{C_n^\lambda(1)} \cos n\theta > 0, \quad -1 \leq x \leq 1, \quad 0 \leq \theta \leq \pi, \quad -1 < r < 1. \quad (4.14)$$

As a corollary of (4.14) Seidel and Szász remark that for a fixed x , $-1 \leq x \leq 1$,

$$\frac{C_n^\lambda(x)}{C_n^\lambda(1)} = \int_0^\pi \cos n\theta \, d\mu(\theta), \quad \lambda > 0, \quad n = 0, 1, \dots \quad (4.15)$$

Here $d\mu(\theta)$ is a positive measure of total mass one which depends on x but is independent of n .

Observe that (4.15) is still another integral representation and that it is not contained in the previous section. Actually, while it isn't contained in our previous work and we can not find $d\mu(\theta)$ explicitly, it is possible to reprove (4.14) using results in Section 2 and even to find some extensions of it.

THEOREM 1. *Let $\lambda > \mu > 0$. Then*

$$\sum_{n=0}^{\infty} \frac{(n + \mu) C_n^\mu(\cos \theta) C_n^\lambda(\cos \phi) r^n}{C_n^\lambda(1)} > 0, \quad -1 < r < 1, \quad 0 \leq \phi, \quad \theta \leq \pi. \quad (4.16)$$

PROOF. Using $C_n^\lambda(\cos(\pi - \phi)) r^n = (-r)^n C_n^\lambda(\cos \phi)$ it is sufficient to prove (4.16) for $0 < \phi \leq \pi/2$. By continuity it is sufficient to show it for $0 < \phi < \pi/2$. Then use (3.13) in (4.16) and we see that we have

$$\int_0^\phi \sum_{n=0}^{\infty} (n + \mu) C_n^\mu(\cos \theta) \frac{C_n^\mu(\cos \psi)}{C_n^\mu(1)} \left(\frac{r \cos \phi}{\cos \psi} \right)^n K(\phi, \psi, \lambda, \mu) \, d\psi \quad (4.17)$$

where $K(\phi, \psi, \lambda, \mu)$ is a nonnegative function which is independent of n . But the sum in (4.17) is positive since $|(r \cos \phi)/(\cos \psi)| \leq |r| < 1$ and (3.49) combined with the convolution structure shows that the series in (4.17) is positive.

COROLLARY 1. *Let $\lambda > \mu > 0$, $-1 \leq x \leq 1$. Then there is a positive measure $d\mu(y)$ such that*

$$\frac{C_n^\lambda(x)}{C_n^\lambda(1)} = \int_{-1}^1 \frac{C_n^\mu(y)}{C_n^\mu(1)} d\mu(y) \quad (4.18)$$

where $\int_{-1}^1 d\mu(y) = 1$ and $d\mu(y)$, while it depends on x , is independent of n .

PROOF. (4.16) is equivalent to

$$r^n \frac{C_n^\lambda(x)}{C_n^\lambda(1)} = \int_{-1}^1 \frac{C_n^\mu(y)}{C_n^\mu(1)} d\mu_{x,r}(y).$$

We let $r \rightarrow 1$ and $d\mu_{x,r}(y) \rightarrow d\mu_x(y)$. If we let $\mu \rightarrow 0$ in Theorem 1 and Corollary 1 we get the Seidel-Szász results.

Using asymptotic formulas for $C_n^\lambda(x)$, $C_n^\mu(y)$ we can say something about the above measure $d\mu(y)$, but we have not been able to find an explicit formula for it. It suffices to say that it is always absolutely continuous except possibly when $x^2 = 1$ or $y^2 = 1$ and the only singularity it has as a function is when $x = y$. There the singularity is the same type that occurs in (3.13), that is, it behaves like $|y - x|^{\lambda-\mu-1} dy$ when $\lambda - \mu < 1$ and x and y are bounded away from ± 1 .

There are many other extensions of (4.14). The following will suffice to show the type of theorem which can be obtained.

THEOREM 2. *If $\alpha_j > \beta_j$, $\alpha_j \geq -1 - \beta_j$, and $-1 < r < 1$, then*

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n \prod_{j=1}^k \frac{P_n^{(\alpha_j, \beta_j)}(\cos \theta_j)}{P_n^{(\alpha_j, \beta_j)}(1)} \cos n\theta > 0. \quad (4.19)$$

PROOF. For $\theta_j = 0, j = 1, 2, \dots, k$, this is just

$$P(r, \theta) = \frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n\theta = \frac{(1-r^2)}{2(1-2r \cos \theta + r^2)} > 0. \quad (4.20)$$

Then using $\cos n\theta \cos n\phi = \frac{1}{2} [\cos n(\theta + \phi) + \cos n(\theta - \phi)]$ we see that

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n \prod_{j=1}^k \cos n\phi_j \cos n\theta > 0. \quad (4.21)$$

From (4.15) and (4.21) we see that

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n \prod_{j=1}^k \frac{C_n^{\lambda_j}(\cos \psi_j)}{C_n^{\lambda_j}(1)} \cos n\theta > 0, \quad \lambda_j \geq 0.$$

This is the same as

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n \prod_{j=1}^k \frac{P_n^{(\lambda_j - \frac{1}{2}, \lambda_j - \frac{1}{2})}(\cos \psi_j)}{P_n^{(\lambda_j - \frac{1}{2}, \lambda_j - \frac{1}{2})}(1)} \cos n\theta > 0. \quad (4.22)$$

Then using Bateman's integral (3.3) we see that

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n \prod_{j=1}^k \frac{P_n^{(\alpha_j, \beta_j)}(\cos \psi_j)}{P_n^{(\alpha_j, \beta_j)}(1)} \cos n\theta > 0 \quad (4.23)$$

for (α_j, β_j) satisfying $\alpha_j \geq \beta_j$, $\alpha_j \geq -1 - \beta_j$.

(4.19) contains the following inequality of Lyness and Moler [24]

$$\sum_{k=0}^{\infty} (-1)^k \left[\frac{\sin(k+1)\varphi}{k+1} \right]^{2l} \geq 0, \quad l = 1, 2, \dots \quad (4.24)$$

For we let $\alpha_j = \beta_j = 1$, $\theta_j = \varphi$, k be $2l$, $\theta = \pi$ and let $r \rightarrow 1$. The series will converge for $l = 1, 2, \dots$, so we may let $r \rightarrow 1$. Actually (4.16) can be extended in the same way for $\mu = 1$ and we obtain

$$\sum_{k=0}^{\infty} (k+1)^2 (-1)^k \left[\frac{\sin(k+1)\varphi}{k+1} \right]^{2l} \geq 0, \quad l = 2, 3, \dots, \quad (4.25)$$

which is much stronger than (4.24). Also we can obtain

$$\sum_{k=0}^N (-1)^k \left[\frac{\sin(k+1)\varphi}{k+1} \right]^{2l} \geq 0, \quad l = 1, 2, \dots, \quad N = 0, 1, \dots \quad (4.26)$$

from (4.1) for $\alpha = \beta = \frac{1}{2}$ and the convolution structure for ultraspherical series. This may have interesting consequences for quadrature problems, for Lyness and Moler came across (4.24) in some work on quadrature problems. A completely trivial proof of (4.24) follows from the argument given above when we observe that the special case of (3.13) which we can use to prove (4.22) is just the integrated form of (3.16).

As in most branches of harmonic analysis whenever a theorem is proved it pays to look for a dual theorem. Most of the results of this chapter have duals and we give three sample theorems. Others will suggest themselves to the reader.

THEOREM 3.

$$\int_0^{\pi} e^{-\epsilon\theta} P_n^{(\alpha, \beta)}(\cos \theta) d\theta > 0, \quad n = 0, 1, \dots, \quad \epsilon > 0, \quad \alpha \geq \beta \geq -\frac{1}{2}.$$

For $\alpha = \beta = -\frac{1}{2}$ this is

$$\int_0^{\pi} e^{-\epsilon\theta} \cos n\theta d\theta = \frac{\epsilon}{\epsilon^2 + n^2} [1 + (-1)^{n+1} e^{-\epsilon\pi}] > 0 \quad (4.27)$$

by a simple computation. Then using (3.44) we see that $\int_0^{\pi} e^{-\epsilon\theta} P_n^{(\beta, \beta)}(\cos \theta) d\theta > 0$ for $\beta \geq -\frac{1}{2}$ and using (3.40) we have

$$\int_0^{\pi} e^{-\epsilon\theta} P_n^{(\alpha, \beta)}(\cos \theta) d\theta > 0, \quad \alpha \geq \beta \geq -\frac{1}{2}.$$

THEOREM 4.

$$\int_0^\phi P_n^{(\alpha, \beta)}(\cos \theta) \sin \theta \, d\theta \geq 0 \quad \text{for} \quad 0 \leq \phi \leq \pi, \quad \alpha \geq \beta \geq \frac{1}{2}.$$

As above it is sufficient to consider $\alpha = \beta = \frac{1}{2}$. Then we must show that $\int_0^\phi \sin(n+1)\phi \, d\phi \geq 0$. But this is just

$$\int_0^\phi \sin(n+1)\phi \, d\phi = \frac{1 - \cos(n+1)\phi}{n+1} \geq 0.$$

We can extend the range of (α, β) at the expense of weakening the theorem for the above values of (α, β) .

THEOREM 5.

$$\int_0^\phi P_n^{(\alpha, \beta)}(\cos \theta) \, d\theta \geq 0 \quad \text{for} \quad 0 \leq \phi \leq \pi, \quad \alpha \geq \beta \geq 0.$$

It is sufficient to prove Theorem 5 for $\alpha = \beta = 0$. Using for $\lambda = \frac{1}{2}$ and $\mu = 1$ we see that

$$\int_0^\phi P_n(\cos \theta) \, d\theta = \sum_{k=0}^{\infty} \alpha_{k,n} \int_0^\phi \sin(n+2k+1)\theta \, d\theta$$

where

$$\alpha_{k,n} = \frac{2(n+k)! \, \Gamma(k + \frac{1}{2})}{\pi k! \, \Gamma(n+k + \frac{3}{2})} > 0.$$

Each of these integrals is nonnegative by the same calculation as above.

A natural question which occurs is which of these theorems corresponds to the problem (4.1). As Szegő [18] pointed out (4.1) for $\alpha = \beta$ implies

$$\int_0^\theta \varphi^{-\alpha} J_\alpha(\varphi) \, d\varphi \geq 0, \quad \alpha \geq 0. \quad (4.28)$$

If we use (3.35) we see that Theorem 5 also implies (4.28), so Theorem 5 is the natural analogue of (4.1). Szegő has examined (4.28) in detail and has shown that it holds for some $\alpha < 0$. As he pointed out, (4.1) seems to be a harder problem and the same holds for Theorem 5. It would be of interest to obtain them for some $\alpha < 0$. There are other integrals like (4.28) which are also positive. However we don't have a complete solution to the problem of when $\int_0^\theta \varphi^{-\beta} J_\alpha(\varphi) \, d\varphi \geq 0$, so we pass over this problem now.

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